Tides and Trigonometry

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1 Purpose

The current picture of how satellites (such as the Moon) raises tides on the body it is orbiting (such as the Earth) requires a complicated mathematical expression. However, the conceptual physics can be reduced to a trigonometry problem suitable for a competent high school student, requiring only the law of sines, the law of cosines, and knowledge of supplementary angles. A sketch of the problem is illustrated in Fig. 1

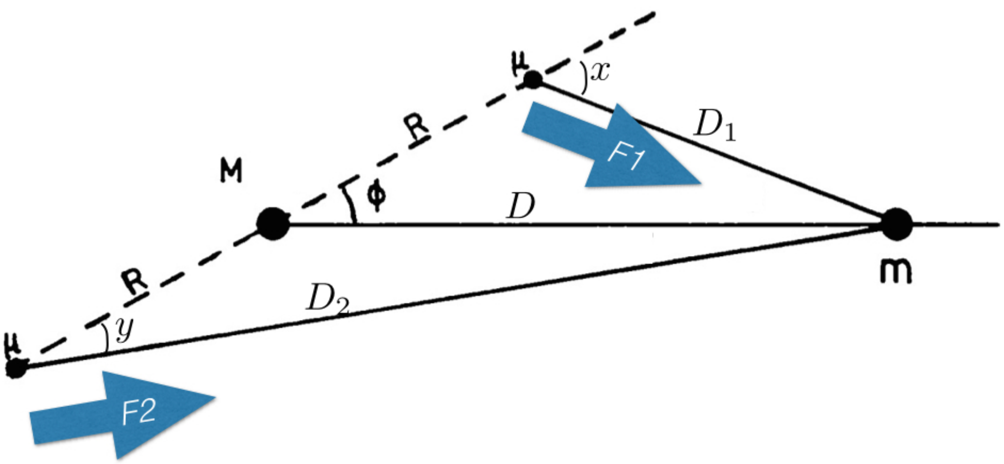

Figure 1: Trigonometry component of solving the tidal problem. The lengths D and R are known (the Earth-Moon distance and the height of tides from the center of the Earth, respectively), as is the angle ϕ. The students must solve for D_1, D_2, and the angles x and y. M is the mass of the Earth, m the mass of the moon, and μ is the mass of the tides (that is, roughly how much mass is caught up in the tides of the ocean). This problem is taken from [1].

This lesson is designed to have students independently determine trigonometric quantities given related lengths and angles. They will then apply their results to determine the amount of time by which a day lengthens every century due to tides.

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2 Overview

Most people are aware that the tides on Earth are caused by the Moon; however, an important addition to this fact is that the tides actually slow down the rotation of the Earth, causing a day to become longer. In this lesson, students will use trigonometry and algebra to determine exactly how much the rotation of the Earth is slowed every century due to the tides from the Moon.

3 Student Outcomes

Students will be able to:

- Calculate lengths and angles of triangles using:
  - Law of Sines,
  - Law of Cosines,
  - Supplementary Angles.
- Input results from the trigonometry problem into an algebraic problem, and
- Conceptually explain how the Moon creates tides on Earth, and how those tides effect the rotation of the Earth.

4 Standards Addressed

The lesson addresses the following Common Core State Standards:

- **CCSS.MATH.CONTENT.HSG.SRT.D.10** - Prove the Laws of Sines and Cosines and use them to solve problems.
- **CCSS.MATH.CONTENT.HSG.SRT.D.11** - Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

5 Time

This lesson can be divided into three sections: the initial section in which the problem is explained to the students (approximately 10 minutes), the independent work section where the students work out the relevant trigonometric quantities from Fig. 1 (approximately 20 minutes), and the final section in which the students use their results to determine how much a day on Earth lengthens per century using the given equations (approximately 10 minutes). This suggests a total time of **40 minutes**.

6 Level

This lesson is designed for students in a High School Trigonometry class who are already familiar with the Law of (Co)Sines.

7 Materials and Tools

- Scientific calculator with sin and cos functions,
- **Tides on Earth** handout (Attached)

8 Preparation

Apart from printing the handout and ensuring that every student has a calculator, no preparation is required.

9 Prerequisites

It is expected that students will be familiar with the law of sines and the law of cosines, and that they can work open-ended problems where a large trigonometry problem is given with quantities to be solved, but the direction or the specific method are left up to the students.

It is also suggested that students be conceptually familiar with the concepts of gravity and that applying a torque to an object changes how it rotates, but these can be explained in the lesson if additional time is allocated.

10 Background

We know that tides on planets (such as the Earth) are caused by the gravity of orbiting satellites (such as the Moon). The basic concept is as follows: the since the gravitational force of an object depends on the distance from it, other objects at different distances will experience different forces. For the Earth and Moon, the Earth is sufficiently large that different sides of the planet are different distances from the Moon. This implies that different sides of the Earth experience a different gravitational pull from the Moon. In particular, the side closest to the Moon experiences a higher gravitational force than the side farthest from the Moon. This difference in forces causes the oceans on the near side to bulge outwards towards to the moon, while the oceans on the far side bulge away from the moon. See Fig. 2

As the Earth rotates, this bulge gets turned away slightly from the direction facing the Moon. The gravitational fore eventually pulls it back, but since the Earth is always rotating, the tidal

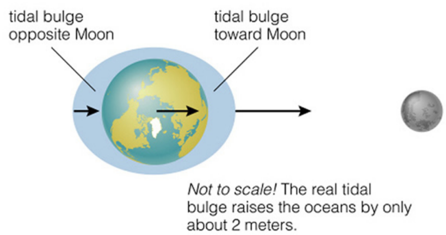


Figure 2: The picture of static tides raised on the Earth by the Moon.

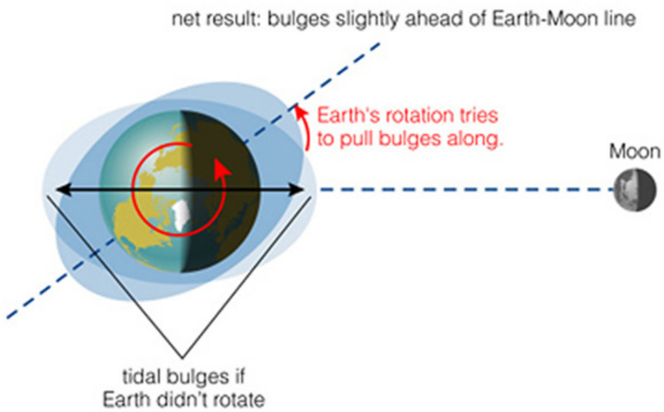


Figure 3: Tidal bulge rotated away from the Moon by the rotation of the Earth.

bulge of the Earth is always offset from directly above the Moon by a small angle, ϕ. See Fig 3.

With this constant bulge present on Earth, the gravitational force on different tides is no longer pointed in the same direction (arrows in Fig. 1). Since the two forces are now working in different directions, they can exert a torque upon the Earth. Over several centuries, this torque slows down the rotation of the earth, making a day longer and longer.

If we know exactly how strong the gravitational pull on different sides of the Earth is, we can work out how much longer a day gets over the course of 100 years.

11 Teaching Notes

When teaching this lesson, it is generally best if the students are allowed to work independently when analyzing Fig 1, as this lets them explore an open-ended problem with techniques they have previously learned. During this phase, the teacher can approach individual students to offer advice and assistance.

At the end of the open-ended section, the students can be invited to write their solution on the board, and compare the results of their individual methods to arrive at the answer.

Note: When determine the angle supplementary to angle x, the students may attempt to use the law of sines with ϕ, R, and D. However, this particular case represents a degenerate case where the law of sines fails. Instead of the true angle, the law of sines will return π minus the true angle. This is because two different triangles can have the given sides and angle. See [2] for more information.

Once the students have a numeric value for the angles x and y, and the distances D_1 and D_2, these values can be used to determine the torque on the Earth and the deceleration of the Earth due to tides. This step is provided as a purely algebraic exercise, with the equations given on the last page of the handout.

12 Assessment

At the simplest level, the effectiveness of this lesson can be assessed by ensuring that the students have computed the correct answer by the end of the exercise. However, a more effective assessment is to approach students during the independent trigonometry section and have them explain to the teacher the technique they are using to solve for the given variables. This will allow the instructor to assess and correct in real time any errors or confusion that the students may be experiencing.

References

[1] P Hut. Tidal evolution in close binary systems. Astronomy and Astrophysics, 99:126–140, June 1981.

[2] Wikipedia. Law of sines: The ambiguous case. http://en.wikipedia.org/wiki/Law_of_sines#The_ambiguous_case, 2014.

Name:______________________________ May 8, 2014

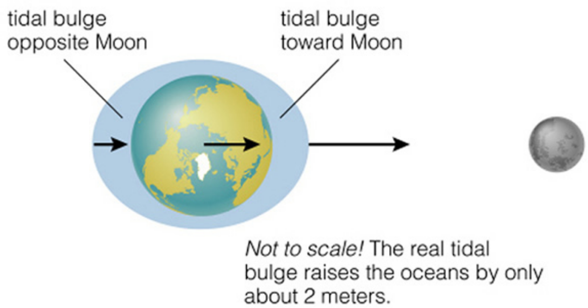
Algebra II/Trig: Tides

1 Tides on Earth

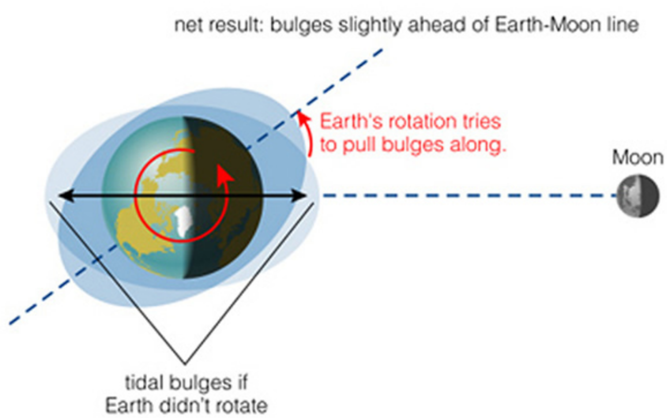
Tides on Earth are caused by the gravitational pull of the moon. Gravity is inversely proportional to the square of the distance between two objects:

$$F = G\frac{mM}{D^2}$$

Since the distance between the moon is not the same on either side of Earth, the force of gravity is greater on the side facing the moon. This difference in gravitational force is what causes tides:



Since the Earth rotates faster on its axis (once per day) than the moon orbits around the Earth (once every 27.3 days), the tides on Earth actually get rotated forward, causing the water to rush ahead of the moon:



Because the two bulges now pull on the moon at different angles, the tides on Earth torque the planet, causing the Earth's rotation to slow down.

2 Modeling Tides

When we talk about tides, we use a simplified model, where we treat each of the tidal bulges as a small mass:

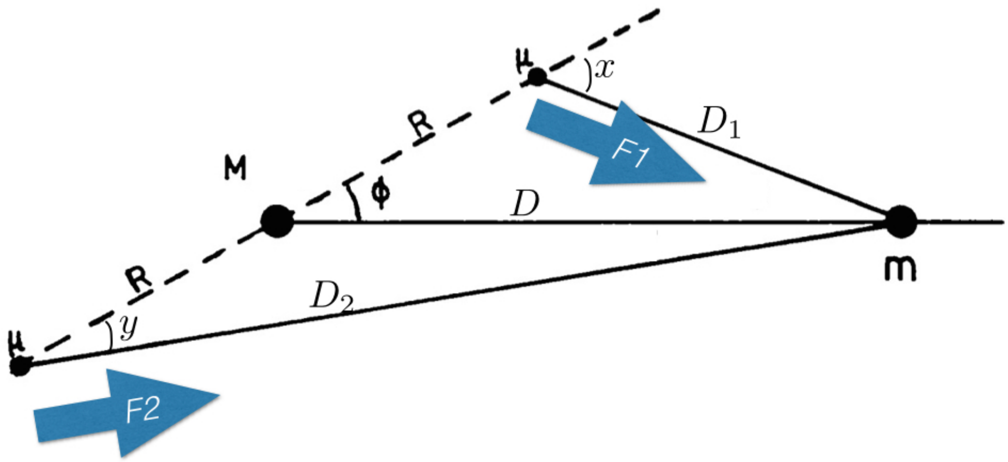


Figure 1: Model for how tides cause forces on Earth (Hut 1981)

We want to know what force the tides exert on the moon. We know that the moon is (on average) 3.844×10^5 km from Earth. We also know from Oceanographers that the surface of the oceans is about 6378.1 km above the center of the Earth, and that the rotation of the Earth pulls the tidal bulge in front of the Earth by about 0.4 degrees.

Question 1

What we need to know is how far each tidal bulge is from the moon, in order to find out what the force of gravity is beween them.

Using whatever technique you want (law of sines, law of cosines, etc), find the distances D_1 and D_2 and the angles x and y

Question 2

We know that the Earth has a mass of 6×10^{24} kg, and the moon has a mass of 7.3×10^{22} kg. If $G = 6.67384 \times 10^{-11} \frac{m^3}{kg \times s^2}$, and the mass of the tides is $\mu = 7.86 \times 10^{12}$ kg, what are $F1$ and $F2$?

Question 3

Finally, use your results, and the following equation, to find α, the rate at which the rotation of Earth is slowing.

$$F1 \sin(x) - F2 \sin(y) = \frac{2}{5} MR\alpha$$

Your answer will be in units of $\frac{1}{s^2}$. Convert it to units of s to find out how much of a day we loose to tides every century. To do that, multiply by

$$\alpha \cdot 4.3 \times 10^{23} s^3 =$$