Interacting Stars: The Gravitational N-body Problem

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Time
One single class period (40-50 minutes)

Level
Appropriate for any high school grade level. Subjects: Astronomy (unit on star clusters or on the use of computers in astronomy); also could be modified for computer science or math.

Purpose
This lesson will introduce students to an example of a problem in astronomy that can only be solved with computers. Computers are used in many different areas of astronomical research today. There are classes of astronomical objects that we would like to model mathematically, but doing so requires the use of computers, because it would just take to long to model by hand. One such type of objects is *star clusters*, which are groups of thousands to millions of stars held together by gravity. The stars interact with each other through the force of gravity, and this drives the cluster as a whole to evolve. In general, this problem is called the *Gravitational N-body problem*, described as follows: *you have N stars, each of which is attracted to every other star because of gravity – how does the system of stars evolve?* The concept of the problem is fairly straightforward. But solving it is actually quite difficult. The stars will move around from the net gravitational forces of the system - the question is, where will the stars be after a certain amount of time? How will the overall structure of the cluster evolve in time? For a real star cluster with thousands to millions of stars, this simply cannot be done by hand. Scientists must turn to computers to solve the N-body problem.

In this activity students will get a sense of how challenging it is to model the evolution of star clusters by figuring out how many mutual gravitational interactions there are among a set of stars, and thus how many calculations are required to get all of the gravitational forces in the system at a given time. They will learn how computers help to make solving this problem feasible.

Students will start with simple examples involving just a few stars, but will eventually have to develop a *general* method for calculating the number of interactions among *any* number of stars (an equation). This procedure of pattern recognition and generalization is an important aspect of student learning. Also, to be able to assess when and how computers can be helpful in solving a complicated problem is one of the cornerstones of *Computational Thinking,* a movement that has gotten a lot of attention in recent years. Furthermore, star clusters are a very active and exciting topic in present-day astronomy research.

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Overview
Introduction

To start, have a discussion with students about the way that computers and computer modeling are used in the world – not just for entertainment, but also for making life easier and for solving problems. Ask them to try to generate their own ideas. Here are some examples you can share: modeling weather to provide you with your daily forecast; storing and organizing your music in your iTunes library; online marketplaces, like Amazon, that allow you to buy everything you need without leaving your home; creation of amazing animated movies, like Shrek.
Then point out that scientists from all disciplines use computers substantially in research – computers and computer programming are not just for computer scientists! Tell the students that they will be exploring one example where computers are essential for solving a real problem in astronomy – the *gravitational N-body problem*, which is the basis of theoretical studies of star clusters.

A point that you should make to the students is *why* one would want to do all the calculations to model a star cluster in the first place – just to make a cool movie? Why can't we just use telescopes to just observe real star clusters? The quick answer is that when we look out at star clusters, we see just a snapshot in its life. But, since star clusters take a very long time to evolve (billions of years), we will never *see* their evolution in real time. Computer simulations allow us to model how star clusters *evolve*, or change over time. We provide the physics (how we think these things work) and we see what happens – can we make model clusters that look like the real ones that we observe? If so, then we gain confidence that our understanding of the physics is on track. Are there some differences between our models and the real star clusters that we observe? Is there a particular piece of the puzzle that we don't quite understand? If so, then there is still more work to be done.

One of the goals of the day will be to see just how difficult it would be to model the evolution of star clusters by hand. While the basic concept of the gravitational interactions of stars is well understood and can, in principle, be computed with pen and paper, to do this for a realistic star cluster of thousands to millions of stars would take a very long time. Computers are good at doing calculations very quickly – one can program a computer to do all of the calculations needed to model a star cluster evolving in time.

Worksheet

Part 1: Students will look at a diagram showing 6 stars. They will figure out how many gravitational interactions there are among the stars, considering that there is a force between each *pair* of stars. They will draw lines between each pair to represent the interaction, and count the lines.

Students can work individually or in small groups. They can read through and work on worksheet on their own. Go around to the different groups to see if students are doing it correctly. A very brief group discussion at the end of part 1 might be useful to make sure everyone is on the same page.

Part 2: This part is a bit more challenging. Students will derive a general expression for determining the number of interactions. Some students may be able to reason through the problem on their own, others will struggle. Students should at least try to figure out the expression on their own or in small groups, but in the end it might be the most useful to discuss this as a group. Some students will probably be able to figure it out on their own, or at least get close, if given a bit of time. It will be interesting to discuss the reasoning of different students, as they may arrive at the answer in different ways.

The desired expression is $N^* (N-1) / 2$ (see teacher notes for more details)

Part 3: Analysis questions can be done individually, either in class or as homework, depending on time. These questions will help students and instructor assess their understanding of the material. They will help students to understand the big picture relating to this activity.

Student Outcomes

Learner Objectives:

- Students can explain the basic idea of the gravitational N-body problem
- Students can explain why this problem is challenging, and how computers can help
- Students can determine the number of interactions among a set of stars

Illinois State Science Standards

12.D.3b Explain the factors that affect the gravitational forces on objects (e.g., changes in mass, distance).

13.A.5c Explain the strengths, weaknesses and uses of research methodologies including observational studies, controlled laboratory experiments, computer modeling and statistical studies.

Next Generation Science Standards (NGSS)

Science and Engineering Practices: *Using Mathematics and Computational Thinking*

- Use mathematical, computational, and/or algorithmic representations of phenomena or design solutions to describe and/or support claims and/or explanations.
- Apply techniques of algebra and functions to represent and solve scientific and engineering problems.
- Use simple limit cases to test mathematical expressions, computer programs, algorithms, or simulations of a process or system to see if a model "makes sense" by comparing the outcomes with what is known about the real world.

Prerequisites:

Students should have an idea of what gravity is – that it is a force that causes two objects with mass to be attracted to one another. Students do not have to know the equation that describes this force, but it would help them understand the goals of the problem.

Background

An understanding of the basic nature of the gravitational force – that there exists a force between all objects with mass – is all that is required for this activity. Since stars have mass, they interact via the gravitational force. You will have to apply mathematical logic to make the jump from analyzing a simple example of a handful of interacting stars, to the more general problem of N interacting stars – all you must do is develop a simple algebraic expression (it turns out to make the problem much simpler!) In the end you will appreciate the complexity of the gravitational N-body problem, and that as N gets larger, the problem becomes more and more difficult to solve. For real star clusters, it is simply impossible to do by hand!

Teaching Notes

Depending on the group of students you have, students may benefit more from either small group work or from a larger group discussion. Students should get through the first part of the activity quickly on their own. They should find that for 6 stars, there are 15 lines/pairs/interactions. For 7 stars, there are 21 lines/pairs/interactions.

The second part will take longer and will require more guidance (at least for some students).

Ask the students for different ways of thinking about the problem, or different ways of obtaining the desired expression. Students think differently, and so they apply their logic in different ways to get the result. Some students may have encountered a problem similar to this in a different context, and others might have more developed mathematical reasoning skills that will help them here.

The expression that will be derived in part 2 is **N* (N-1) / 2**

There are different ways to think about the problem. One way may be a bit more straightforward than the other, but note that different students may reason through it in different ways.

Easier way: Consider star 1 - it interacts with N-1 other stars – that makes **(N-1)** interactions. The same goes for stars 2, 3, etc., so we get a factor of N out front: that gives us **N*(N-1)**. But, we already counted the interaction 1-2 when we considered star 1; when we considered star 2, we counted its interaction with star 1 again! The same for 2-3, 3-4, etc. We want to consider just ONE interaction per pair (since there is just *one* gravitational *force* between the pair, and thus *one* calculation). To avoid double counting each interaction, we *divide by 2* at the end to get the final expression

N* (N-1) / 2

More difficult way (maybe): Consider star 1 – it interacts with stars 2, 3, 4, 5, 6 – that makes 5, or N-1, interactions. Now consider star 2 – we already counted its interaction with star 1, so we only have to count the interactions with stars 3, 4, 5 and 6 – that gives 4 more interactions. Consider star 3 – it interacts with 4, 5 – makes 3 more interactions, etc. So we get 5+4+3+2+1=15 interactions. This logical route might come more easily to the students, but it is not as useful for arriving at the formula above that will work for even vary large numbers.

You may want to have some numbers ready to share with the students, or to use to test their calculations along the way. What if there are a million stars – how many interactions are there? If you assume that it takes a minute to calculate the gravitational force between two stars (which is reasonable), how many years would it take a person to calculate the forces among a set of a million stars?

Pre-class Preparation
Calculate ahead of time any numbers that you would like to provide for the students (or have them calculate), such as an estimate of how long it would take a single person to calculate all of the gravitational forces between a thousand as well as a million stars. Also consider how you will help the students arrive at the equation for calculating the number of pairs of interactions for a set of stars, in case the students run into trouble coming up with it themselves.

Materials and Tools
Computer and projector for displaying movie
Student handouts (provided)
Exit Slip/quiz (can be graded or not)

Assessment
In addition to going around to students/groups while they work on the activity, some means of informal assessment will be useful. I suggest that, when deriving the expression for the number of interactions, you do the following: Ask students if they understand (the expression, the logic, etc) - they can give you a thumbs up (if they completely understand), thumbs sideways (if they partly understand, but are not completely comfortable), or thumbs down (if they are lost).

There are analysis questions at the end of the packet, which can be graded or just assessed informally to ensure that students are meeting the objectives.

There is an *exit slip* with three questions relating to the activity and the learning objectives. This can either be given at the end of the period, or at the beginning of the next class period.

Question 1 addresses students overall understanding of the topic of the day – the gravitational N-body problem – what it is, what it relates to, and why it is challenging.

Question 2 and 3 address whether the student can apply their mathematical logic to figure out the number of interactions among a set of stars. In question 2, the number is small – students might draw a picture, and draw and count the lines representing the pair-wise interaction; OR they may use the formula that was derived. In question 3, students can't simply draw lines to figure out the answer – here you will see how well students understood the derivation of the formula.

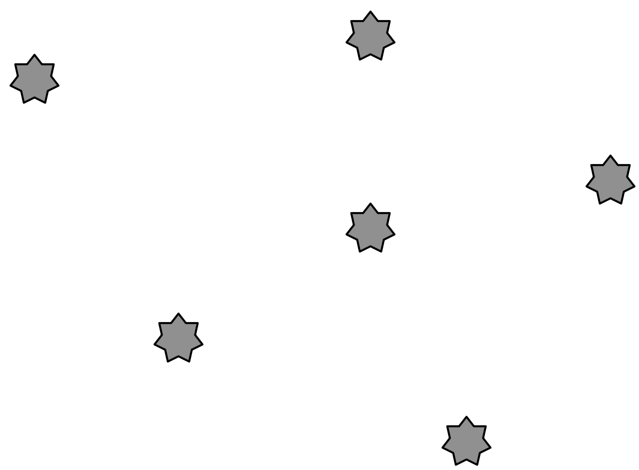
Handouts begin on following page

Interacting Stars and the Gravitational N-body Problem

How many calculations would it take?

Star clusters are groups of stars that are all held together by the force of gravity. They can have anywhere from thousands up to millions of stars! In star clusters, stars interact with each other through the force of gravity. We can learn about how star clusters change over time by studying these interactions. But for large star clusters with up to millions of stars, this can be challenging. If we wanted to calculate all of the gravitational forces between every pair of stars in a star cluster at a given instant in time, *how many calculations would that be?*

Part 1: Let's start with a simple example. Consider a very small star cluster with just 6 stars, as shown in the diagram below. Answer the questions below (feel free to draw on the diagram).



1. If every *pair* of stars interacts with each other, how many interactions are there in total? ____

2. Add one more star to the diagram to make *seven* stars in total.

    a. How many *additional* interactions are there when you add the 7th star? ____

    b. How many interactions are there in *total* for 7 stars? ____

3. Imagine adding one more star. Do you see a pattern in how the *number of interactions* increases as the *number of stars* increases? Explain the pattern that you observe.

Part 2: Real star clusters have *thousands* to *millions* of stars. Would you want to calculate the number of interactions in a real star cluster the same way you did in the example with 6 stars? It would take quite a long time! But if we can understand the pattern of how the *number of interactions* changes with the *number of stars*, then maybe we can come up with an *equation* to tell us how many interactions there will be for ANY number of stars.

3. Write a *mathematical expression* that allows you to calculate the number of interactions among *N* stars, where *N* could be ANY number. Test your expression for values of *N* for which you *already know the number of interactions* (to make sure that it actually works).

4. Choose an *actual number* for *N* that would be a reasonable estimate for the number stars in a real star cluster. Using your expression from question 3, calculate the number of interactions among the stars.

5. How long would it take to calculate that many interactions by hand? To do this, you must estimate how long it would take you to calculate the gravitational force between a single pair of stars, which would involve plugging numbers into the equation $F = G\, m_1\, m_2 / r^2$.

Analysis Questions

Q1. This topic in astronomy is called the *Gravitational N-body problem*. Explain in your own words why you think this name was chosen.

Q. How does the number of interactions *scale* with the number of stars in the system? For example, does it scale *linearly* (the number of interactions is proportional to the number of stars, N)? Does it increase faster than linearly? Determine the scaling relationship between the number of interactions and the number of stars. Why does this task of calculating all the forces by hand get so difficult for large numbers of stars?

Q. Why are computers helpful in this scenario (modeling the evolution of star clusters, solving the gravitational N-body problem)?

EXIT SLIP

1. Explain the "gravitational N-body problem," list what kinds of systems in astronomy relate to this kind of study, and explain why computers are needed.

2. If there are 5 stars, how many gravitational interactions are there among them? Show how you came up with the answer.

3. If there are 100 stars, how many gravitational interactions are there? What if there are N stars?